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EXPERIMENTAL ASTRONOMY LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE 39, MASSACHUSETTS

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Experimental Astronomy Laboratory
Massachusetts Institute of Technology
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Approved Winston R. Markey
Director (R.G.S.)
Experimental Astronomy Laboratory

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CONSERVATION LAWS AND LIAPOUNOV STABILITY OF THE FREE ROTATION OF A RIGID BODY

John P. Vinti

M.I.T. Experimental Astronomy Laboratory, Cambridge, Massachusetts

ABSTRACT

The paper derives the well known stabilities of free rotation of a rigid body about its principal axes of least and greatest moments of inertia directly from the constancy of the kinetic energy and of the square of the angular momentum. The resulting proof of Liapounov stability yields new quantitative measures of this stability. Involving only simple algebra, it depends on the fulfilling of a weak sufficient condition that insures an unchanging sign of the main component of the angular velocity $\underset{\sim}{\omega}$. The method cannot be used, however, to prove the well known instability of rotation about the intermediate axis.

The quantitative results for the radii of the spheres in $\underset{\sim}{\omega}$-space that occur in the Liapounov proof lead to a physical result that may be of interest. If the Earth were truly a rigid body, rotating freely, the angular deviation of its instantaneous polar axis from the nearest principal axis could not increase from a given initial value by more than the factor $\sqrt{2}$.

These same quantitative results for the radii of the Liapounov spheres in $\underset{\sim}{\omega}$-space also lead to sufficient conditions for the rotational stability of almost spherical bodies of various shapes, prolate or oblate. They may be pertinent in designing "spheres" to be used in currently planned experiments to test general relativity by observing the rate of precession of a rotating sphere in orbit about the Earth.

1. Integrals of the Motion.

Let A, B, and C be the moments of inertia of a rigid body about its principal axes, with A<B<C, let $1_A, 1_B$, and 1_C denote unit vectors attached to those axes, and let ω_1, ω_2, and ω_3 be the respective components of the body's angular velocity about those axes.

Then, for free motion or for motion in a uniform gravitational field, its angular momentum

$$\underset{\sim}{L} = 1_A A\omega_1 + 1_B B\omega_2 + 1_C C\omega_3 \tag{1}$$

and its kinetic energy of rotation T, given by

$$A\omega_1{}^2 + B\omega_2{}^2 + C\omega_3{}^2 = 2T \tag{2}$$

both taken relative to the center of mass as origin, are constant. From (1) there also follows:

$$A^2\omega_1{}^2 + B^2\omega_2{}^2 + C^2\omega_3{}^2 = \underset{\sim}{L}^2 = \text{const} \tag{3}$$

This note shows that the integrals of motion (2) and (3) guarantee Liapounov stability of rotation about either of the axes 1_A or 1_C, without recourse to the differential equations of motion. It also shows, however, that these integrals, alone, do not suffice to prove Liapounov instability of rotation about the intermediate axis 1_B.

2. The Cases where the Initial Angular Velocity $\underset{\sim}{\omega}$ is Strictly along One of the Principal Axes.

If one sets the time derivative of $\underset{\sim}{L}$ equal to zero, one obtains the usual Eulerian first-order differential equations. If the initial $\underset{\sim}{\omega}(0)$ is equal to any of $1_A\omega_1(0)$, $1_B\omega_2(0)$, or $1_C\omega_3(0)$, it then follows that $\underset{\sim}{\omega}(t)$ remains equal to this initial value.

Let us attempt to obtain these results from (2) and (3) alone, putting

$$\omega_k(0) \equiv \omega_{k0}, \quad (k=1,2,3).$$

First multiply (2) by A and subtract the result from (3), to obtain

$$B(B-A)\omega_2^2+C(C-A)\omega_3^2=B(B-A)\omega_{20}^2+C(C-A)\omega_{30}^2 \tag{4}$$

Next multiply (2) by C and subtract (3) from the result, to obtain

$$A(C-A)\omega_1^2 + B(C-B)\omega_2^2= A(C-A)\omega_{10}^2 + B(C-B)\omega_{20}^2 \tag{5}$$

Finally, multiply (2) by B and subtract (3) from the result, to obtain

$$A(B-A)\omega_1^2-C(C-B)\omega_3^2 = A(B-A)\omega_{10}^2-C(C-B)\omega_{30}^2 \tag{6}$$

Since $A<B<C$, all terms in (4) and (5) are positive. Thus, from (4), if $\omega_{20}=\omega_{30}=0$, it follows that $\omega_2(t)=\omega_3(t)=0$ for all t. Similarly, from (5), if $\omega_{10}=\omega_{20}=0$, it follows that $\omega_1(t)=\omega_2(t)=0$ for all t. On the other hand, if the initial rotation is about the intermediate axis, so that $\omega_{10}=\omega_{30}=0$, (6) does not show that $\omega_1(t)=\omega_3(t)=0$, although the statement happens to be true.

Eqs. (4) and (5) show that the graphs of ω_2 vs. ω_3 or of ω_1 vs. ω_2 are ellipses and Eq.(6) shows that the graph of ω_1 vs. ω_3 is a hyperbola. These facts suggest some conclusions that are well known: with respect to small changes in the initial conditions, the rotation is stable if it is about 1_A or 1_C, but unstable if it is about 1_B (MacMillan 1966).

3. Liapounov Stability

These conclusions about stability are old and the free motion of a rigid body has had exhaustive treatments in the literature, analytically with elliptic functions and geometrically with Poinsot ellipsoids, polhodes, and herpolhodes. It is of some interest, however, to demonstrate the stability of free rotation about l_A or l_C solely from the constancy of $\underset{\sim}{L}^2$ and T and to show that it is of the Liapounov type (L-type). In this way we obtain quantitative measures of the stability.

The contrast with the motion of a particle is important. Particle motion is described by a differential system of the 6th order. In that case Liapounov stability of equilibrium means this: if the particle is displaced slightly from a point P of equilibrium in the (metricized) six-dimensional phase space, the phase point will always remain inside a six-dimensional sphere of radius ϵ and center P whenever the initially displaced phase point lies inside some like-centered six-sphere of radius $\delta \leqq \epsilon$. Examples are few, e.g., the linear oscillator and motion about a triangular Lagrange point L_4 in the restricted problem of three bodies (Pollard, 1966, Deprit and Deprit-Bartholomé, 1967). In view of the revival of interest in L-stability by the Lagrangian L_4 result, it may be useful to show explicitly that rotational stability is of the Liapounov type. In this case, instead of phase space, we deal with 3-dimensional $\underset{\sim}{\omega}$-space. There is then Liapounov stability of rotational equilibrium at $\underset{\sim}{\omega} = \underset{\sim}{\omega}_i$ if there exists a $\delta \leqq \epsilon$ such that when $\underset{\sim}{\omega}_i$ is changed by $\delta\underset{\sim}{\omega}_i$, the resulting $\underset{\sim}{\omega}(t)$ always satisfies $|\underset{\sim}{\omega}(t)-\underset{\sim}{\omega}_i| < \epsilon$ if $|\delta\underset{\sim}{\omega}_i| < \delta$.

4. Free Rotation about l_A

Since $(\omega_0, 0, 0)$ is an equilibrium point in the $\underset{\sim}{\omega}$-space, any motion near it is thus L-stable if there exists a $\delta \leqq \epsilon$ such that

$$(\omega_1-\omega_0)^2+\omega_2{}^2+\omega_3{}^2<\epsilon^2 \tag{7}$$

whenever

$$(\omega_{10}-\omega_0)^2+\omega_{20}^2+\omega_{30}^2 < \delta^2$$

With no loss of generality, we may take $\omega_1(0) \equiv \omega_{10} = \omega_0$, since it is only the non-vanishing of ω_{20} and ω_{30} that produces any departure from rotation about 1_A alone. Then (8) becomes

$$\omega_{20}^2+\omega_{30}^2 < \delta^2 \tag{9}$$

Now, from (4)

$$\omega_2^2-\omega_{20}^2 = \frac{C}{B}\,\frac{C-A}{B-A}\,(\omega_{30}^2-\omega_3^2) \tag{10}$$

so that

$$\omega_2^2-\omega_{20}^2 < \frac{C}{B}\,\frac{C-A}{B-A}\,\omega_{30}^2 \tag{11}$$

and

$$\omega_3^2 < \omega_{30}^2 + \frac{B}{C}\,\frac{B-A}{C-A}\,\omega_{20}^2 \tag{12}$$

From (5), with $\omega_{10}=\omega_0$, we find

$$\omega_0^2-\omega_1^2 = \frac{B}{A}\,\frac{C-B}{C-A}\,(\omega_2^2-\omega_{20}^2) \tag{13}$$

$$< \frac{B}{A}\,\frac{C-B}{C-A}\,\omega_2^2 \tag{14}$$

Insertion of (11) into (14) gives

$$\omega_0^2-\omega_1^2 < \frac{B}{A}\,\frac{C-B}{C-A}\,\omega_{20}^2 + \frac{C}{A}\,\frac{C-B}{B-A}\,\omega_{30}^2 \tag{15}$$

This relation gives no information about the sign of ω_1, which we must now consider. Note especially that if ω_1 changes sign, $(\omega_1-\omega_0)^2$ will at some time exceed ω_0^2 and (7) cannot be fulfilled for $\varepsilon<|\omega_0|$. Let us therefore impose the condition that ω_1 shall not change sign. To do so, note that if it does, it must go through $\omega_1=0$, in which case (15) would become

$$\omega_0^2 < \frac{B}{A}\,\frac{C-B}{C-A}\,\omega_{20}^2 + \frac{C}{A}\,\frac{C-B}{B-A}\,\omega_{30}^2 \tag{16}$$

If we now deny (16), ω_1 cannot change sign and we shall not have ruled out stability. Thus

$$\omega_0^2 > \frac{B}{A}\,\frac{C-B}{C-A}\,\omega_{20}^2 + \frac{C}{A}\,\frac{C-B}{B-A}\,\omega_{30}^2 \tag{17}$$

is a sufficient condtion for no change of sign of ω_1.

To place an upper bound on $(\omega_1-\omega_0)^2$ in (7), we need first an upper bound on $|\omega_1^2-\omega_0^2|$. To obtain it, note first that

$$\omega_1^2-\omega_0^2 < \frac{B}{A}\frac{C-B}{C-A}\omega_{20}^2 , \tag{18}$$

by (13). If we now insert (10) into (13), we also find

$$\omega_1^2-\omega_0^2 = \frac{C}{A}\frac{C-B}{B-A}(\omega_3^2-\omega_{30}^2) \tag{19}$$

$$> -\frac{C}{A}\frac{C-B}{B-A}\omega_{30}^2 \tag{20}$$

From (18) and (20) it follows that $|\omega_1^2-\omega_0^2|$ is less than either of $\frac{B(C-B)}{A(C-A)}\omega_{20}^2$ and $\frac{C}{A}\frac{C-B}{B-A}\omega_{30}^2$ and therefore less than their sum[(1)], so that

$$|\omega_1^2-\omega_0^2| < \frac{B}{A}\frac{C-B}{C-A}\omega_{20}^2 + \frac{C}{A}\frac{C-B}{B-A}\omega_{30}^2 \tag{21}$$

Now

$$|\omega_1-\omega_0| \equiv \frac{|\omega_1^2-\omega_0^2|}{|\omega_0+\omega_1|} \tag{22}$$

Also, since $\omega_1(0)\equiv\omega_{10}\equiv\omega_0$, it follows that $|\omega_0+\omega_1|>|\omega_0|$, if we impose the sufficient condition (17) that ω_1 does not change sign. Then

$$(\omega_1-\omega_0)^2 < \frac{(\omega_1^2-\omega_0^2)^2}{\omega_0^2} \tag{23}$$

by (22).

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1. Of course it is actually less than half their sum. If, however, we should thereby attempt to sharpen the proof, we should find formulae connecting ϵ and δ that would not guarantee the fulfilling of (17) whenever $\epsilon < |\omega_0|$.

But (21) gives an upper bound for $(\omega_1^2-\omega_0^2)^2$ and (17) for $1/\omega_0^2$. Insertion of these bounds into (23) then gives

$$(\omega_1-\omega_0)^2 < \frac{B}{A}\frac{C-B}{C-A}\,\omega_{20}^2 + \frac{C}{A}\frac{C-B}{B-A}\,\omega_{30}^2 \tag{24}$$

From (11),(12), and (24) it then follows that

$$(\omega_1-\omega_0)^2+\omega_2^2+\omega_3^2<\omega_{20}^2\left[1+\frac{B}{A}\frac{C-B}{C-A}+\frac{B}{C}\frac{B-A}{C-A}\right]_1+\omega_{30}^2\left[1+\frac{C}{A}\frac{C-B}{B-A}+\frac{C}{B}\frac{C-A}{B-A}\right]_2 \tag{25}$$

Since $A<B<C$, it follows that $[\ ]_1 < [\ ]_2$, so that

$$(\omega_1-\omega_0)^2+\omega_2^2+\omega_3^2<(\omega_{20}^2+\omega_{30}^2)\left[1+\frac{C}{A}\frac{C-B}{B-A}+\frac{C}{B}\frac{C-A}{B-A}\right] \tag{26}$$

Thus we have

$$(\omega_1-\omega_0)^2+\omega_2^2+\omega_3^2< \epsilon^2 \tag{27}$$

if

$$\omega_{20}^2+\omega_{30}^2 < \delta^2 \tag{28}$$

where

$$\delta^2 = \epsilon^2\left[1+\frac{C}{A}\frac{C-B}{B-A}+\frac{C}{B}\frac{C-A}{B-A}\right]^{-1} < \epsilon^2, \tag{29}$$

provided that

$$\omega_0^2 > \frac{B}{A}\frac{C-B}{C-A}\,\omega_{20}^2 + \frac{C}{A}\frac{C-B}{B-A}\,\omega_{30}^2 \tag{17}$$

With the very permissive assumption that $\epsilon^2< \omega_0^2$, the relations (28) and (29) guarantee the fulfillment of the proviso (17), so that the latter may be dropped as an explicit condition. To show this, note that if $\epsilon^2<\omega_0^2$, (28) and (29) lead to

$$u_1 \equiv \left(1+\frac{C}{A}\frac{C-B}{B-A}+\frac{C}{B}\frac{C-A}{B-A}\right)(\omega_{20}^2+\omega_{30}^2) < \omega_0^2 \tag{30}$$

Then

$$\frac{C}{A}\,\frac{C-B}{B-A}\,(\omega_{20}^{2}+\omega_{30}^{2}) < u_1 < \omega_0^{2} \tag{31}$$

But from A<B<C,

$$\frac{B}{A}\,\frac{C-B}{C-A} < \frac{C}{A}\,\frac{C-B}{B-A}\,, \tag{32}$$

so that

$$\frac{B}{A}\,\frac{C-B}{C-A}\,\omega_{20}^{2} + \frac{C}{A}\,\frac{C-B}{B-A}\,\omega_{30}^{2} < \omega_0^{2}\quad , \tag{33}$$

which is (17). This proves the statement.

Then for arbitrarily small ϵ, the $\underset{\sim}{\omega}$-point will always remain inside a sphere of radius $\epsilon < |\omega_0|$ with center $(\omega_0,0,0)$, if it lies initially inside a like-centered sphere of radius

$$\delta = \epsilon\left[1+\frac{C}{A}\,\frac{C-B}{B-A}+\frac{C}{B}\,\frac{C-A}{B-A}\right]^{-\frac{1}{2}} \tag{34}$$

Thus the free rotation about the axis of smallest moment of inertia is Liapounov-stable. Note, however, that if A=B, so that the C-axis is an axis of symmetry, δ vanishes. Thus for a body with an axis of symmetry, the proof of Liapounov stability fails for rotation about any axis perpendicular to the axis of symmetry.

5. Free Rotation about 1_C

If the $\underset{\sim}{\omega}$-point is initially at $(0,0,\omega_0)$, it remains unchanged. Suppose now that the initial point is changed to $(\omega_{10},\omega_{20},\omega_0)$.

Multiply (2) by C and subtract (3) from the result, to obtain

$$\omega_1^{2}-\omega_{10}^{2} = \frac{B}{A}\,\frac{C-B}{C-A}\,(\omega_{20}^{2}-\omega_2^{2}), \tag{35}$$

from which

$$\omega_1^{2} < \omega_{10}^{2} + \frac{B}{A}\,\frac{C-B}{C-A}\,\omega_{20}^{2} \tag{36}$$

and

$$\omega_2^2 < \omega_{20}^2 + \frac{A}{B}\frac{C-A}{C-B}\,\omega_{10}^2 \tag{37}$$

Next multiply (2) by A and subtract the result from (3), to find

$$\omega_0^2-\omega_3^2 = \frac{B}{C}\frac{B-A}{C-A}\,(\omega_2^2-\omega_{20}^2), \tag{38}$$

where $\omega_3(0) \equiv \omega_{30} = \omega_0$. Then

$$\omega_0^2-\omega_3^2 < \frac{B}{C}\frac{B-A}{C-A}\,\omega_2^2 \tag{39}$$

Insertion of (37) into (39) then shows that

$$\omega_0^2-\omega_3^2 < \frac{B}{C}\frac{B-A}{C-A}\,\omega_{20}^2 + \frac{A}{C}\frac{B-A}{C-B}\,\omega_{10}^2 \tag{40}$$

As before, if ω_3 changes sign, it must go through the value zero, so that if it does

$$\omega_0^2 < \frac{B}{C}\frac{B-A}{C-A}\,\omega_{20}^2 + \frac{A}{C}\frac{B-A}{C-B}\,\omega_{10}^2 \tag{41}$$

Thus

$$\omega_0^2 > \frac{B}{C}\frac{B-A}{C-A}\,\omega_{20}^2 + \frac{A}{C}\frac{B-A}{C-B}\,\omega_{10}^2 \tag{42}$$

is a sufficient condition that ω_3 shall not change sign. Now by (38)

$$\omega_3^2-\omega_0^2 < \frac{B}{C}\frac{B-A}{C-A}\,\omega_{20}^2 \tag{43}$$

Also, by inserting (37) into (38), we find

$$\omega_3^2-\omega_0^2 > -\frac{A}{B}\frac{C-A}{C-B}\,\omega_{10}^2 \tag{44}$$

Thus $|\omega_0^2-\omega_0^2|$ is less than either of $\frac{B}{C}\frac{B-A}{C-A}\,\omega_{20}^2$ and $\frac{A}{B}\frac{C-A}{C-B}\,\omega_{10}^2$, so that it less than their sum. Accordingly

$$|\omega_0{}^2-\omega_3{}^2| < \frac{B}{C}\,\frac{B-A}{C-A}\,\omega_{20}^2 + \frac{A}{B}\,\frac{C-A}{C-B}\,\omega_{10}^2 \tag{45}$$

With use of

$$|\omega_0-\omega_3| \equiv \frac{|\omega_0{}^2-\omega_3{}^2|}{|\omega_0+\omega_3|} \tag{46}$$

and imposition of the condition (42), which makes $|\omega_0+\omega_3|>|\omega_0|$ we have

$$(\omega_0-\omega_3)^2 < \frac{(\omega_0{}^2-\omega_3{}^2)^2}{\omega_0{}^2} \tag{47}$$

Then (42),(45), and (47) yield

$$(\omega_0-\omega_3)^2 < \frac{B}{C}\,\frac{B-A}{C-A}\,\omega_{20}^2 + \frac{A}{C}\,\frac{B-A}{C-B}\,\omega_{10}^2 \tag{48}$$

Addition of (36), (37), and (48) then gives

$$\omega_1{}^2+\omega_2{}^2+(\omega_3-\omega_0)^2 < \omega_{20}^2\left[1+\frac{B}{C}\,\frac{B-A}{C-A}+\frac{B}{A}\,\frac{C-B}{C-A}\right]_3$$
$$+\omega_{10}^2\left[1+\frac{A}{C}\,\frac{B-A}{C-B}+\frac{A}{B}\,\frac{C-A}{C-B}\right]_4 \tag{49}$$

The condition A<B<C is not sufficient to tell whether $[\;]_3$ or $[\;]_4$ is the larger. We therefore rewrite (49) as

$$\omega_1{}^2+\omega_2{}^2+(\omega_3-\omega_0)^2<\omega_{10}^2+\omega_{20}^2+\omega_{20}^2\left(\frac{B}{C}\,\frac{B-A}{C-A}+\frac{B}{A}\,\frac{C-B}{C-A}\right)+\omega_{10}^2\left(\frac{A}{C}\,\frac{B-A}{C-B}+\frac{A}{B}\,\frac{C-A}{C-B}\right) \tag{50}$$

Then, since $\omega_{20}^2 < \delta^2$ and $\omega_{10}^2 < \delta^2$ if $\omega_{10}^2+\omega_{20}^2<\delta^2$, we have

$$\omega_1{}^2+\omega_2{}^2+(\omega_3-\omega_0)^2 < \epsilon^2\,, \tag{51}$$

if

$$\omega_{10}^2+\omega_{20}^2 < \delta^2\,, \tag{52}$$

where

$$\delta^2=\epsilon^2\left[1+\frac{B}{C}\,\frac{B-A}{C-A}+\frac{B}{A}\,\frac{C-B}{C-A}+\frac{A}{C}\,\frac{B-A}{C-B}+\frac{A}{B}\,\frac{C-A}{C-B}\right]^{-1} \tag{53}$$

Again, with the assumption $\epsilon^2 < \omega_0^2$, the relations (52) and (53) guarantee the fulfillment of the proviso (42), so that the latter may be dropped, as an independent requirement. To show this, note that if $\epsilon^2 < \omega_0^2$, (52) and (53) lead to

$$u_3 \equiv \left(1 + \frac{B}{C}\frac{B-A}{C-A} + \frac{B}{A}\frac{C-B}{C-A} + \frac{A}{C}\frac{B-A}{C-B} + \frac{A}{B}\frac{C-A}{C-B}\right)(\omega_{10}^2 + \omega_{20}^2) < \epsilon^2 < \omega_0^2 \tag{54}$$

Then

$$\frac{B}{C}\frac{B-A}{C-A}\omega_{20}^2 + \frac{A}{C}\frac{B-A}{C-B}\omega_{10}^2 < u_3 < \omega_0^2 \tag{55}$$

But this is simply (42), so that the statement is true. Then, for arbitrarily small ϵ, the $\underset{\sim}{\omega}$-point will always remain inside a sphere of radius $\epsilon < |\omega_0|$ with center $(0,0,\omega_0)$, if it lies initially inside a like-centered sphere of radius

$$\delta = \epsilon\left[1 + \frac{B}{C}\frac{B-A}{C-A} + \frac{B}{A}\frac{C-B}{C-A} + \frac{A}{C}\frac{B-A}{C-B} + \frac{A}{B}\frac{C-A}{C-B}\right]^{-\frac{1}{2}} \tag{56}$$

Thus free rotation about the axis of largest moment of inertia is Liapounov-stable.

6. Free Rotation about the Intermediate Axis 1_B

If the $\underset{\sim}{\omega}$-point is initially at $(0,\omega_0,0)$, it remains unchanged. If it is initially at $(\omega_{10},\omega_0,\omega_{30})$, the motion is unstable. The constancy of $\underset{\sim}{L}^2$ and T, however, is not sufficient to demonstrate this instability.

To see why, note that the constancy of $\underset{\sim}{L}^2$ and T applies for all values of the time t, including those values so close to t=0 that $\omega_2 \approx \omega_0$, $\omega_1 \approx \omega_{10}$, and $\omega_3 \approx \omega_{30}$. That is, any conclusion that we can deduce from the constancy of $\underset{\sim}{L}^2$ and T will have to hold also for values of

$$R^2 \equiv (\omega_2-\omega_0)^2 + \omega_1^2 + \omega_3^2 \approx \omega_{10}^2 + \omega_{30}^2 \tag{57}$$

which are arbitrarily small for arbitrarily small values of ω_{10}^2 and ω_{30}^2.

On the other hand, to show L-instability, we should have to show that there exists a positive number that R(t) can exceed as t increases, no matter how small a value we choose for a non-vanishing R(0). Clearly we cannot do so by using relations, such as the constancy of $\underset{\sim}{L}^2$ and T, that are independent of the time.

7. Necessary Conditions for L-Stability

If we could derive necessary conditions for L-stability from the constancy of $\underset{\sim}{L}^2$ and T, we should simply have to deny them to obtain sufficient conditions for L-instability. We have just seen, however, that we cannot derive sufficient conditions for L-instability from the constancy of $\underset{\sim}{L}^2$ and T. Therefore we cannot derive necessary conditions for L-stability from the constancy of $\underset{\sim}{L}^2$ and T.

8. Applications

(a) Stability of Rotation about the 1_A Axis

Let

$$A/C \equiv 1 - \eta_1 \tag{58.1}$$

$$B/C \equiv 1 - \eta_2 \tag{58.2}$$

Then from (34)

$$\delta^2 = \epsilon^2 \left[1 + (\eta_1 - \eta_2)^{-1} \left(\frac{\eta_2}{1-\eta_1} + \frac{\eta_1}{1-\eta_2} \right) \right]^{-1} < \frac{\epsilon^2}{2} \tag{59}$$

If we consider the case of a uniform oblate spheroid, for which A=B<C, initially rotating about an axis 1_A perpendicular to its axis of symmetry, then $\eta_1=\eta_2$ and $\delta=0$. We cannot then prove stability of rotation; the result is in agreement with our knowledge that the rotation is unstable.

If we now consider the case of a uniform prolate spheroid, for which A<B=C, initially rotating about its axis 1_A of symmetry, then $\eta_2=0$ and

Thus

$$\frac{\alpha}{\alpha_i} = \left(\frac{\omega_1^2+\omega_2^2}{\omega_{10}^2+\omega_{20}^2}\right)^{\frac{1}{2}} \tag{68}$$

For A=B, however, (50) gives

$$\omega_1^2+\omega_2^2 < 2(\omega_{10}^2+\omega_{20}^2) \tag{69}$$

Insertion of (69) into (68) then shows that

$$\alpha < \alpha_i\sqrt{2} \tag{70}$$

To estimate the effects of axial asymmetry of the Earth, note that the theory of potential gives

$$\frac{B-A}{4\left[C-\frac{A+B}{2}\right]} = J_2^{-1}(C_{2,2}^2+S_{2,2}^2)^{\frac{1}{2}} \approx 10^{-3} \tag{71}$$

Here J_2 is the coefficient of the second zonal harmonic in the spherical harmonic expansion of the Earth's gravitational potential. $C_{2,2}$ and $S_{2,2}$ are the coefficients of sectorial harmonic terms. Thus

$$\frac{B}{C}\frac{B-A}{C-A} \approx \frac{A}{C}\frac{B-A}{C-B} \approx (4)10^{-3} \tag{72}$$

as compared with

$$\frac{B}{A}\frac{C-B}{C-A} \approx \frac{A}{B}\frac{C-A}{C-B} \approx 1, \tag{73}$$

so that by (50) axial asymmetry cannot change the factor 2 in (69) or the factor $\sqrt{2}$ in (70) by more than a few percent, even if we allow for some error in the estimate 10^{-3} in (71).

If the Earth were truly a rigid body, rotating freely, the angular deviation of its instantaneous polar axis from the nearby principal axis could then not increase from a given initial value by more than the factor $\sqrt{2} \approx 1.4$. A mere glance at the figure for polar wandering on p.34 of Vol. 1, S.A.O. Special Report 200 (1966) shows that the actual polar motion is very different. The polar wandering is clearly much greater than that which could occur if the principal axis were fixed and the above factor $\sqrt{2}$ were valid for the actual earth. The conclusions of this paper are thus compatible with and reenforce the idea that one must invoke non-rigidity to explain polar wandering.

The present paper had its origin in discussions with Dr. S. J. Madden, Jr., who first thought of using Eqs. (4) through (6) to investigate stability of rotation. Dr. André' Deprit also contributed some very helpful comments.

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NOTATION

A	smallest principal moment of inertia
B	intermediate principal moment of inertia
C	largest principal moment of inertia
$C_{2,2}$	a sectorial coefficient in the spherical harmonic expansion of the Earth's gravitational potential
$\underset{\sim}{L}$	total angular momentum of rotation
J_2	coefficient of the second zonal harmonic of the Earth's potential
T	kinetic energy of rotation
$S_{2,2}$	a sectorial coefficient in the expansion of the Earth's potential
u_1	abbreviation used only in (30) and (31)
u_3	abbreviation used only in (54) and (55)
α	angle between Earth's instantaneous axis of rotation and the nearest principal axis
α_i	an initial value of α
α_f	a final value of α
ϵ	an arbitrarily small radius in $\underset{\sim}{\omega}$-space
δ	a small radius in $\underset{\sim}{\omega}$-space ($\delta<\epsilon$)
η_1	$1-A/C$
η_2	$1-B/C$
$\underset{\sim}{\omega}(t)$	angular velocity of rotation at time t
$1_A, 1_B, 1_C$	unit vectors along the principal axes
ω_1	component of $\underset{\sim}{\omega}$ along 1_A
ω_2	component of $\underset{\sim}{\omega}$ along 1_B
ω_3	component of $\underset{\sim}{\omega}$ along 1_C
ω_{k0}	$\omega_k(0)$
ω_0	value of $\omega_k(0)$, when considering stability of rotation about the k'th principal axis